Signed graphs, regular matroids, grafts

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draft

Signed Graphs - Regular Matroids - Grafts

by

A.M.H. Gerards and A. Schrijver

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by A.M.H. Gerards[1)] and A. Schrijver[1,2)]

Abstract

We exploit the theory of regular matroids to study nice classes of signed graphs (i.e. undirected graphs with odd and even edges) and of grafts (i.e undirected graphs with odd and even nodes, associated with T-joins). These classes are: signed graphs with no odd-K_4 and no odd-K_3^2, and grafts with no K_4-partition and no $K_{3,2}$-partition (odd-K_4 and odd-K_3^2, are special types of signed graphs, K_4-partition and K_3^2- partition are special types of grafts). We give a constructive characterization of these classes, using Seymour's decomposition theorem for regular matroids. Moreover we derive characterizations from the orientability of a regular matroid. The latter characterizations we use to formulate several optimization problems related to odd cycles in signed graphs with no odd-K_4 and no odd-K_3^2 and to T-joins in grafts with no K_4-partition and no $K_{3,2}$-partition as min-cost-circulation problems. As a consequence we prove some well-known min-max relations due to Seymour for these optimization problems. We also show how some graph theoretic results follow.

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1. Introduction

A signed graph is a pair (G,E_0), where $G = (V(G),E(G))$ is an undirected graph and E_0 is a subset of the edge set $E(G)$ of G. We allow multiple edges and loops in G. The edges in E_0 are called odd, the other edges even. A cycle C in G is called odd (even) if $E_0 \cap E(C)$ is odd (even, respectively). A signed graph is bipartite if it contains no odd cycles. In this paper a central role is played by the signed graphs indicated in figure 1. Wriggled and dotted lines stand for (pairwise openly disjoint) paths, dotted lines may have length zero, and odd indicates that the corresponding faces are odd cycles. Each signed graph of the first type is called an odd-K_4, each signed graph of the second type an odd-K_3^2.

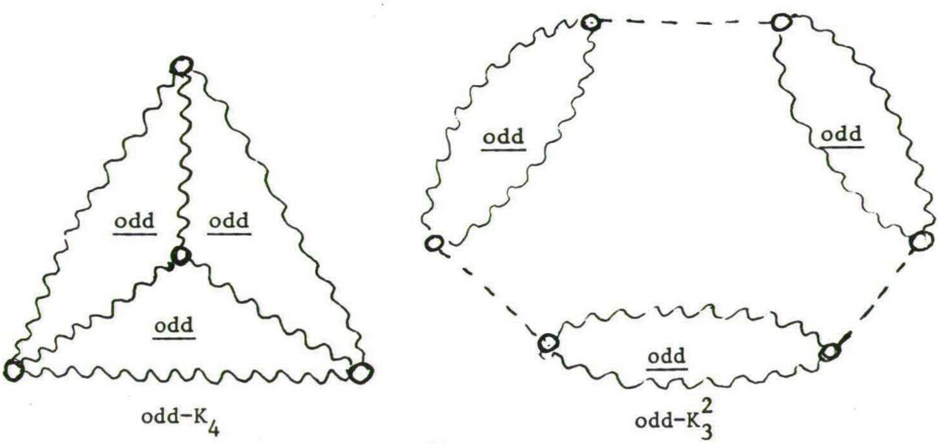


figure 1

In this paper we list a number of characterizations of those signed graphs which do not contain an odd-K_4 or an odd-K_3^2 as a subgraph. Mainly, these characterizations follow from the theory of regular matroids: in section 2 we define for each signed graph (G,E_0) an associated binary matroid, $M(G,E_0)$. It turns out that (G,E_0) does contain no odd-K_4 and no odd-K_3^2 if and only if $M(G,E_0)$ is regular. In the subsequent sections we exploit results on regular matroids obtained by Tutte and Seymour (section 3, 4 and 5). In section 6 we give a characterization of signed graphs not containing an odd-K_4. The final section, section 7, we discuss a different object. Seymour [1980] introduces in his paper on the decomposition of regular matroids the concept of grafts, i.e pairs [G,T] where G is an undirected graph and T is a subset of V(G). In parallel with the sections 2, 3, 4 and 5 we give characterizations of those grafts for which a certain associated binary matroid is regular.

2. Preliminaries; Binary Matroids Associated with Signed Graphs

Let G be an undirected graph, and let M_G be its node-edge incidence matrix, i.e. M_G is an $V(G) \times E(G)$ matrix with entries 0 and 1. An entry of M_G is 1 if and only if its row index $v \in V(G)$ is an endpoint of its column index $e \in E(G)$. Moreover for $E_0 \subset E(G)$, let $\chi_{E_0} \in \mathbb{R}^{E(G)}$ denote the characteristic vector of E_0 as a subset of $E(G)$.
The matroid $M(G,E_0)$ associated to the signed graph (G,E_0) is the binary matroid represented over GF(2) by the columns of the matrix:

$$\left[\begin{array}{c|c} 1 & \chi_{E_0} \\ \hline 0 & \\ & M_G \\ 0 & \end{array}\right] \tag{2.1}$$

The element of $M(G,E_0)$ not in $E(G)$ (corresponding to the first column of (2.1)), will be denoted by p. The reader will easily deduce the circuits, bases and rank-function of $M(G,E_0)$. With some exceptions throughout the text we use notation and terminology of matroid theory as given in the book of Welsh [1976]. For convenience we use the term circuit for a minimal dependent set in a matroid, and cycle for the familiar subject in a graph. (So a cycle in G is a circuit in $M(G)$, the cycle matroid of G.) Obviously $M(G,E_0) = M(G,E_0 \Delta B)$ for any minimal cut (co-cycle) B of G. (Δ denotes symmetric difference). We call the operation: $E_0 \to E_0 \Delta B$, <u>resigning</u>. We say that (G,E_0) <u>reduces to</u> (G',E_0') if (G', E_0') can be obtained from (G,E_0) by a series of the following operations:

- deleting an edge from G (and from E_0);
- contracting an even edge of G;
- resigning.

The relation of reduction with matroid minors is obvious:
("/" means "deletion", "\" means "contraction")

- $M(G,E_0) \setminus e = M(G \setminus e, E_0 \setminus \{e\})$ if $e \in E(G)$;
- $M(G,E_0)/e = M(G/e, E_0 \Delta B)$ in case $e \in E(G)$ and e is no loop, where $B = \emptyset$ if $e \notin E_0$, and B is any cut of G containing e in case $e \in E_0$;
  If e is an even loop: $M(G,E_0)/e = M(G \setminus e, E_0)$.

If e is an odd loop then $M(G,E_0)/e \cong M(G,E_0)/p$.
(since then e is parallel with p).

To be complete:

- $M(G,E_0) \setminus p$ is the binary matroid with as circuits the even cycles in (G,E_0) and the sets of the form $E(C_1) \cup E(C_2)$ where C_1 and C_2 are odd cycles and $|V(C_1) \cap V(C_2)| \leq 1$.

  $M(G,E_0)/p = M(G)$, i.e. the cycle matroid of the undirected graph G.

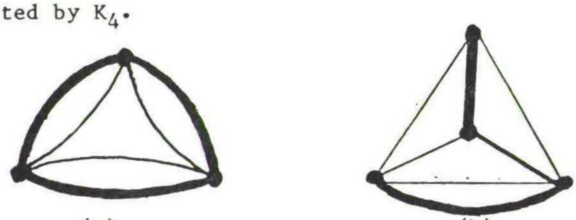
Regular Matroids

For the definition of a regular matroid we refer to Tutte [1971] or Welsh [1976, p. 173]. Tutte [1958] proved that a binary matroid is regular if and only if it does not contain F_7 or F_7^* as a minor. (The binary representation of F_7 and of F_7^* are in figure 2; Welsh [1976] uses the notation M(Fano), M^*(Fano) respectively.)

$$F_7: \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix} \qquad F_7^*: \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$

figure 2

The signed graph in figure 3a will be denoted by K_3^2 (bold edges odd). The signed graph (G,E_0) with G equal to the 4-clique and all edges odd, will be denoted by K_4.



(a) (b)

figure 3

The following propositions are easy to proof:

<u>Proposition 2.2.</u>

Let (G,E_0) be a signed graph. Then:

(i) $M(G,E_0) \cong F_7$ if and only if $(G,E_0) \cong K_3^2$;

(ii) $M(G,E_0) \cong F_7^*$ if and only if $(G,E_0) \cong K_4$ (possibly after resigning).

<u>Proposition 2.3.</u>

Let (G,E_0) be a signed graph.

(i) The following are equivalent:

- $M(G,E_0)$ has an F_7 minor using p;
- (G,E_0) reduces to K_3^2.

(ii) The following are equivalent:

- $M(G,E_0)$ has an F_7^* minor using p;
- (G,E_0) reduces to K_4;
- (G,E_0) contains an odd-K_4.

Note that the assertions in (i) are not equivalent to "(G,E_0) contains an odd-K_3^2". Since the signed graph in figure 3b reduces to K_3^2, but does not contain an odd-K_3^2. However the following does hold:

<u>Proposition 2.4.</u>

Let (G,E_0) be a signed graph. Then (G,E_0) does contain an odd-K_4 or an odd-K_3^2 if and only if (G,E_0) can be reduced to K_4 or to K_3^2 .

The following lemma brings the signed graphs with no odd-K_4 and no odd-K_3^2 within the theory of regular matroids.

<u>Lemma 2.5.</u>

Let (G,E_0) be a signed graph. Then (G,E_0) contains no odd-K_4 and no odd-K_3^2 if and only if $M(G,E_0)$ is a regular matroid.

<u>Proof</u>:

To prove the equivalence we may assume G to be 2-connected. Moreover we may assume that (G,E_0) is not bipartite, and has no even loops. Hence $M(G,E_0)$ is a connected matroid. However for connected matroids Seymour [1977a] extended Tutte's result to: Let x be an element in a connected matroid M. Then M is regular if and only if M has no F_7 minor and no F_7^* minor using x. Together with Propositions 2.3 and 2.4 this proves the lemma (take x = p). □

In section 6 we discuss signed graphs with no odd-K_4. The following result due to Lovász and Schrijver [1985] makes it possible to use results on signed graphs with no odd-K_4 and no odd-K_3^2 to signed graphs with no odd-K_4.

Theorem 2.6. (Lovász, Schrijver [1985])
Let (G,E_0) be a signed graph, satisfying the following property:

(*) If $\{u,v\} \subset V(G)$ separates G, then one side of this two node cutset consists of two parallel edges, e_1 and e_2 say, with $e_1 \in E_0$, $e_2 \notin E_0$, or one side of this two node cutset is bipartite.

Then the following holds:
Let (G,E_0) contain no odd-K_4. Then $(G,E_0) \cong K_3^2$ or (G,E_0) contains no odd-K_3^2.

Proof
Let (G,E_0) be a signed graph satisfying (*). Suppose (G,E_0) contains no odd-K_4, but does contain an odd-K_3^2. Let $(\tilde{G},\tilde{E}_0)$ be an odd-K_3^2 contained in (G,E_0) such that $|E(P_1)| + |E(P_2)| + |E(P_3)|$ is minimal. (P_1, P_2 and P_3 are the paths indicated in figure 4.)

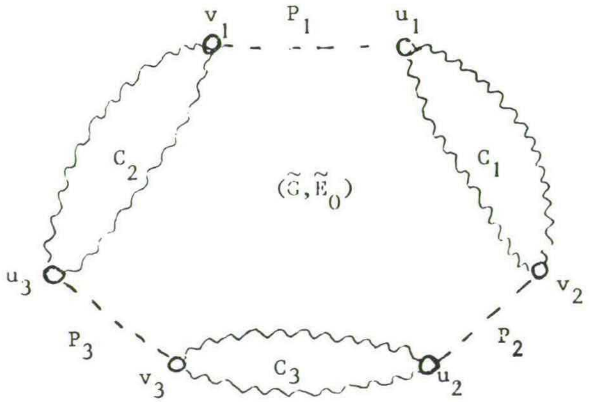


figure 4

The odd cycles C_1, C_2 and C_3, as well as the nodes v_1, v_2, v_3, u_1, u_2, and u_3 are as indicated in figure 4. (Note that v_i may be equal to u_i $(i=1,2,3)$.)

Define: $V_i := V(P_i) \cup V(C_i)$ $(i=1,2,3)$. If $S \subset V(G)$, then a path P from u to v is called an <u>S-path</u> if $V(P) \cap S = \{u,v\}$.

<u>Claim</u>: If P is a $V(\tilde{G})$-path, then P is a V_i-path, for $i=1,2$ or 3.

<u>Proof of claim.</u>

Let P be a $V(\tilde{G})$-path. Let u and v be the endpoints of P. Assume P is no V_i-path $(i=1,2,3)$. Hence we may assume $v \notin \{v_1, v_2, v_3\}$. Moreover we may assume $v \notin V_2$. So $u \notin \{v_2, v_3\}$. Finally we may assume $u \in V_1$. (Indeed, if $u \notin V_1$, then $u \neq v_1$. Interchanging u and v, and renumbering indices yields $u \in V_1$, $v \in V_2$.) We consider three cases.

<u>Case I</u>: $v \in V(C_2) \setminus \{u_2\}$. Then $\tilde{G}$ and P together contain an odd-K_4. This yields a contradiction.

<u>Case II</u>: $u \in V(P_1)$; $v \in V(P_2)$. Then $\tilde{G}$ and P together contain an odd-K_3^2 with smaller $|E(P_1)| + |E(P_2)| + |E(P_3)|$. Again we have a contradiction.

<u>Case III</u>: $u \in V(C_1) \setminus \{u_1\}$, $v \in V(P_2)$. Now there are two possibilities. If the cycle C (see figure 5) is odd then $\tilde{G}$ and P together contain an odd-K_4. If C is even we find an odd-K_3^2 with smaller $|E(P_1)| + |E(P_2)| + |E(P_3)|$. So both possibilities yield a contradiction.

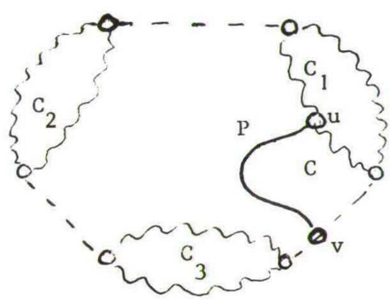


figure 5

<u>end of proof of claim</u>

Since G satisfies (*), the claim yields for $i = 1, 2, 3$: $E(P_i) = \emptyset$, and C_i consists of two parallel edges, one odd and one even. So $(\tilde{G}, \tilde{E}_0) \cong K_3^2$. If $V(G) = V(\tilde{G})$ then by (*): $(G,E_0) = (\tilde{G}, \tilde{E}_0) \cong K_3^2$ and the theorem is proved. So let us suppose: $V(G) \neq V(\tilde{G})$. Let $v \in V(G)\setminus V(\tilde{G})$. By (*) there are three internally node disjoint paths Q_1, Q_2 and Q_3 each going from v to a different node on $\tilde{G}$. But this is impossible since then $\tilde{G}$, Q_1, Q_2 and Q_3 together contain an odd-K_4. □

<u>Remark:</u>

The following result is in a sense dual to Theorem 6.2:

> Let (G,E_0) be a signed graph, which does not reduce to K_3^2. If G is 3-connected then $(G,E_0) \cong K_4$ (possibly after resigning) or (G,E_0) contains no odd-K_4.

The proof essentially relies on the following statements:

- If G is 3-connected then so is $M(G,E_0)$.
- A 3-connected binary matroid with no F_7-minor is regular or equal to F_7^* (Seymour [1980]).
- If an element x of a binary matroid M is not contained in an F_7-minor of M, and M/x has no F_7-minor, then M has no F_7-minor at all.

3. Min-Max Relations

If S is a finite set, $\mathcal{S}$ a collection of subsets of S, and w an integer valued valued function on S, then a w-packing with elements of $\mathcal{S}$ is a family $S_1, S_2, \ldots, S_k$ of members of $\mathcal{S}$ (repetition allowed) such that for each $s \in S$ we have that $|\{i=1,\ldots,k \mid s \in S_i\}| \leq w(s)$. The number k is called the cardinality of the packing.
Seymour [1977b] proved the following result:

Theorem 3.1.
Let M be a binary matroid, and let x be an element of M. Then the following are equivalent:
(i) M does not contain an F_7-minor using x.
(ii) For each weight function w on the elements of M with non-negative integer values, the minimum weight of any set $C \setminus \{x\}$, where C is a circuit of M containing x, is equal to the maximum cardinality of a w-packing with sets of the form $C^* \setminus \{x\}$, where C^* is a cocircuit of M containing x. □

Together with Proposition 2.3, Seymour's result implies:

Corollary 3.2.
Let (G,E_0) be a signed graph.
(i) The following are equivalent:
- (G,E_0) does not contain an odd-K_4.
- For each weight function w: $E(G) \to \mathbb{Z}_+$, we have:
  The maximum cardinality of a w-packing with odd cycles is equal to the minimum weight of a subset of E(G) meeting each odd cycle.

(ii) The following are equivalent:
- (G,E_0) does not reduce to K_3^2.
- For each weight function w: $E(G) \to \mathbb{Z}_+$, we have:
  The minimum length of an odd cycle is equal to the maximum cardinality of a w-packing with subsets of E(G), each meeting each odd cycle. □

So we have a first characterization for signed graphs with no odd-K_4 and no odd-K_3^2.

<u>Corollary 3.3.</u>

Let (G,E_0) be a signed graph. Then (G,E_0) does not contain an odd-K_4 or an odd-K_3^2 if and only if for each weight function $w\colon E(G) \longrightarrow \mathbb{Z}_+$ both min-max relations in Corollary 3.2 hold. □

<u>Remark</u>

Corollary 3.3 can also be derived from the fact that each regular matroid has a totally unimodular (standard) representation matrix (over $\mathbb{Z}$) (Tutte [1958].)

4. Decomposition

In this section we elaborate that every signed graph with no odd-K_4 and no odd-K_3^2 can be decomposed into smaller such signed graphs, or in one of three simple types. Here we use the famous result of Seymour on the decomposition of regular matroids (Seymour [1980]), for the case of signed graphs yielding a decomposition in signed graphs with no odd-K_4 and no odd-K_3^2.

Theorem 4.1. (Seymour [1980])

Let M be a regular matroid, then at least one of the following holds

(1) There exist subsets X_1, X_2 partitioning the element set X of M such that $r_M(X_1) + r_M(X_2) = r_M(X) + k-1$
where $k = 1$, 2 and $|X_1|, |X_2| \geq k$
or $k = 3$ and $|X_1|, |X_2| \geq 6$.

(2) M is graphic, or is cographic, or is equal to the matroid, called R_{10}, represented over $GF(2)$ by the columns of the matrix:

$$\begin{bmatrix} 1&0&0&0&0&1&1&1&0&0 \\ 0&1&0&0&0&0&1&1&1&0 \\ 0&0&1&0&0&0&0&1&1&1 \\ 0&0&0&1&0&1&0&0&1&1 \\ 0&0&0&0&1&1&1&0&0&1 \end{bmatrix}. \qquad \square$$

Remark: Seymour [1980] states his result slightly different: In (1) he only requires: $|X_1|$, $|X_2| \geq 4$ if $k = 3$. However using the statements (7.4), (9.2), and (14.2) of his paper one can sharpen this to: $|X_1|, |X_2| \geq 6$ if $k = 3$. We use this in proving Theorem 4.3.

Important in the decomposition for signed graphs with no odd-K_4 and no odd-K_3^2 is the notion of so-called splits.

Assume E_1, E_2 are nonempty subsets of $E(G)$ partitioning $E(G)$. Denote the set of nodes in $V(G)$ spanned by E_1 and E_2 respectively, by V_1 and V_2 respectively. $\bar{G}_i$ is defined by $V(\bar{G}_i) := V_i$, $E(\bar{G}_i) = E_i$ $(i=1,2)$.

1-split: Let $|V_1 \cap V_2| \le 1$. Then $(\bar{G}_1, E_1 \cap E_0)$ and $(\bar{G}_2, E_2 \cap E_0)$ are said to form a 1-split of (G, E_0).

2-split: Let $|V_1 \cap V_2| = 2$, $V_1 \cap V_2 = \{u,v\}$ say.
Moreover, let for $i = 1, 2$, $\bar{G}_i$ be connected and not a signed subgraph of the signed graph in figure 6.

figure 6

Define (G_1, E_{01}) as follows: If $(\bar{G}_2, E_2 \cap E_0)$ is not bipartite add to $(\bar{G}_1, E_1 \cap E_0)$ the two edges in figure 6. If $(\bar{G}_2, E_2 \cap E_0)$ is bipartite, add a single edge e from u to v. Take $e \in E_{01}$ if and only if there exists an odd uv-path in G_2 (a path is odd if it contains an odd number of odd edges). (G_2, E_{02}) is defined analogously. Now (G_1, E_{01}) and (G_2, E_{02}) are said to form a 2-split of (G, E_0). (In figure 7 we give an example of a 2-split in case $(G_i, E_i \cap E_0)$ is not bipartite for $i=1,2$. The bold edges are odd, the thin edges even.)

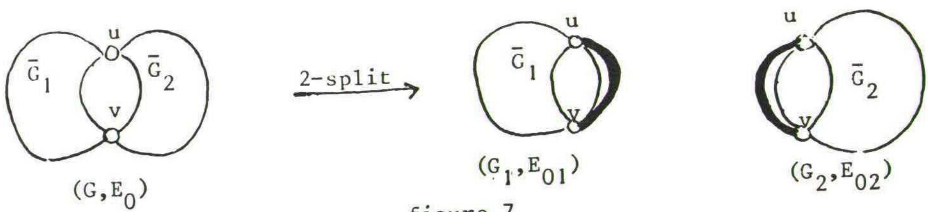


figure 7

3-split: Let $|V_1 \cap V_2| = 3$, $V_1 \cap V_2 = \{u_1, u_2, u_3\}$ say.
Moreover, let $\bar{G}_2$ be bipartite and connected. Finally, let $|E_2| \ge 4$.

G_1 is defined as follows: $V(G_1) := V_1 \cup \{\tilde{v}\}$ (where $\tilde{v}$ is a new node), and $E(G_1) := E_1 \cup \{u_1\tilde{v}, u_2\tilde{v}, u_3\tilde{v}\}$. $\tilde{E}$ is the subset of $\{u_2\tilde{v}, u_3\tilde{v}\}$ defined by: $u_i\tilde{v} \in \tilde{E}$ if and only if there exists an odd path from u_1 to u_i in $(\bar{G}_2, E_2 \cap E_0)$ $(i=2,3)$. We define $E_{01} := (E_1 \cap E_0) \cup \tilde{E}$. Now (G_1, E_{01}) is said to form a 3-<u>split</u> of (G, E_0).

If none of the above assumptions hold we say that no split exists. Note that a 3-split consists of only one signed graph. Moreover note that if no ℓ-split exists for $\ell < k$ $(k=1,2,3)$ then each member of a k-split is a reduction of (G,E_0). The following lemma is easy to prove.

<u>Lemma 4.2.</u>

Let (G,E_0) be a signed graph with a k-split $(k \leqq 3)$ and with no ℓ-split for any $\ell < k$. Then (G,E_0) has no odd-K_4 and no odd-K_3^2 if each part of the k-split has no odd-K_4 and no odd-K_3^2. □

Next we arive at the main result of this section.

<u>Theorem 4.3.</u>

Let (G,E_0) be a signed graph, with no odd-K_4 and no odd-K_3^2. Then at least one of the following holds:

(i) (G,E_0) has a 1-, 2-, or 3-split.

(ii) There exists a node $v_0 \in V(G)$ such that all odd cycles in (G,E_0) contain v_0.

(iii) G is planar with at most two odd faces.

(iv) (G,E_0) is the signed graph in the figure below (possibly after resigning). (Thin edges are even, bold edges are odd.)

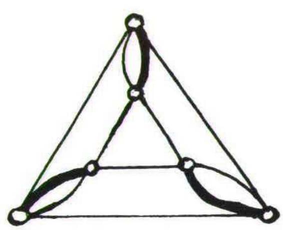


figure 8

Proof:

Let (G,E_0) be a signed graph with no odd-K_4 and no odd-K_3^2. Suppose (G,E_0) has no 1-, 2-, or 3-split. Since $M(G,E_0)$ is regular (Lemma 2.5) we can apply Seymour's theorem (Theorem 4.1). We shall devide the proof into two parts: In part (1) we consider case (1) of Theorem 4.1, and in part (2) we consider case (2).

Part (1): Suppose there exist subsets E_1, E_2 partitioning the edge set of G such that

$$(*)\ r_{M(G,E_0)}(E_1) + r_{M(G,E_0)}(E_2 \cup \{p\}) = r_{M(G,E_0)}(E(G) \cup \{p\}) + k-1$$

with k=1,2 and $|E_1| \geqq k$, $|E_2| + 1 > k$, or k=3 and $|E_1| \geqq 6$, $|E_2| + 1 > 6$. For each $E' \subset E(G)$ we have

$$r_{M(G)}(E') + 1 = r_{M(G,E_0)}(E' \cup \{p\}) = \begin{cases} r_{M(G,E_0)}(E') & \text{if } E' \text{ is not bipartite} \\ r_{M(G,E_0)}(E') + 1 & \text{if } E' \text{ is bipartite} \end{cases}$$

Let $\varepsilon := 0$ of E_1 is bipartite, and $\varepsilon := 1$ if E_1 is not bipartite. Then (*) is equivalent to:

$$(**)\ r_{M(G)}(E_1) + r_{M(G)}(E_2) = r_{M(G)}(E(G)) + (k-\varepsilon) - 1$$

If $|E_2| = 0$, then $k < |E_2| + 1 = 1$. Moreover by (**): $\varepsilon = 0$. Hence (G,E_0) is bipartite, so (iii) holds. So we may assume $|E_2| > 1$.

Let $E_1^1,\ldots,E_1^s;\ E_2^1,\ldots,E_2^t$ be the components of E_1, E_2 respectively. Define the undirected graph H as follows. $V(H) = \{u_1,\ldots,u_s, v_1,\ldots,v_t\}$; for each $v \in V(G)$ spanned by E_1^i and E_2^j there is an edge from u_i to v_j in H $(i=1,\ldots,s;\ j=1,\ldots,t)$. So H may have parallel edges.

Claim 1: $|E(H)| = s + t + k - \varepsilon - 2 = |V(H)| + k - \varepsilon - 2$.

Proof of claim 1: Let V_i be the set of nodes spanned by E_i $(i=1,2)$. Then $r_{M(G)}(E_1) = |V_1| - s$, $r_{M(G)}(E_2) = |V_2| - t$ and $r_{M(G)}(E) =$

$|V(G)| - 1$ (G is connected since G has no 1-split). Since $|V_1 \cap V_2| = |E(H)|$ and $|V_1 \cup V_2| = |V(G)|$, (**) yields the claim.

end of proof of claim 1.

Claim 2: H is a bipartite, connected graph, without isthmuses.

Proof of claim 2: By definition H is bipartite. If H is disconnected, or has an isthmus, then (G,E_0) has a 1-split.

end of proof of claim 2.

Claim 3: H has no two adjacent nodes of degree 2.

Proof of claim 3:

Let u_i, v_j be adjacent nodes of H, both of degree 2. If between u_i and v_j there are parallel edges, then by claim 2: $V(H) = \{u_i, v_j\}$. So $i=j=s=t=1$. By claim 1: $k - \varepsilon = 2$. Now, since (G,E_0) has no 2-split, E_1 or E_2 is contained in the signed graph of figure 6. But since E_1 and E_2 both are connected this means $r_{M(G)}(E_1) + r_{M(G)}(E_2) = r_{M(G)}(E(G))$. So by (**): $k - \varepsilon = 1$, a contradiction. Therefore between u_i and v_j there is only one edge in H. Now $\tilde{E}_1 := E_1^i \cup E_2^j$ and $\tilde{E}_2 := E(G) \setminus \tilde{E}_1$ define a 2-split of G, contradicting our assumption that no 2-split exists.

end of proof of claim 3.

Claim 4: $k = 3$, $\varepsilon = 0$ and H is the graph in figure 9(c) below.

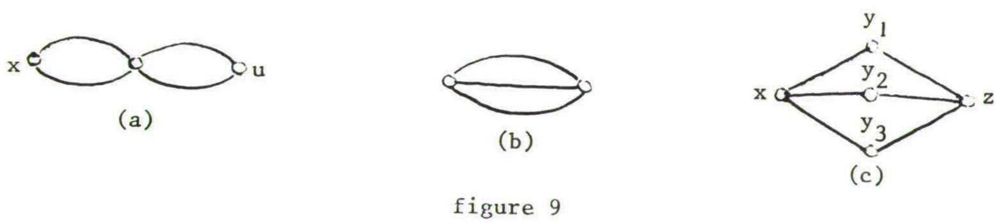


figure 9

Proof of claim 4:

By claims 2 and 3: $|E(H)| > |V(H)| + 1$. Hence by claim 1 $k - \varepsilon - 2 > 1$. So $k = 3$ and $\varepsilon = 0$, and $|E(H)| = |V(H)| + 1$. Using the previous claims, it is easy to see that (a), (b), and (c) (figure 9) are the only possibili-

ties left. It remains to show that H cannot be equal to the graph in figure 9(a) and (b). Since $k = 3$ we have $|E_1| \geq 6$, $|E_2| \geq 5$.
If H is equal to the graph in figure 9(a) then eigther x or y corresponds to an E_1^i or E_2^i with at least three edges. So we would have a 2-split, a contradiction. If H is equal to the graph in figure 9(b) we have a 3-split ($\varepsilon = 0$, so E_1 is bipartite), again a contradiction.

end of proof of claim 4.

We investigate the case that H equals the graph in figure 9(c). If y_1, y_2 and y_3 correspond to E_1^1, E_1^2 and E_1^3 respectively, then we have a 2-split. Indeed, at least one of the E_1^i has cardinality at least 2 (as $|E_1| \geq 6$), and hence it is not contained in the signed graph of figure 6 (as E_1 is bipartite). So y_1, y_2 and y_3 correspond to E_2^1, E_2^2 and E_2^3 respectively, and x and z correspond to E_1^1 and E_1^2. Since (G,E_0) has no 3-split, both $|E_1^1|$ and $|E_1^2|$ are at most 3. But $|E_1| \geqq 6$, and hence $|E_1^1| = |E_1^2| = 3$. Moreover both E_1^1, and E_1^2 are triangles, since otherwise (G,E_0) would have a 2-split. For the same reason each of E_2^1, E_2^2, and E_2^3 is contained in the graph of figure 6. Conclusion: (G,E_0) is contained in the signed graph of figure 7. If (G,E_0) is properly contained in it then possibility (iii) of the theorem holds, and if not then (iv) holds. Summarizing, we have seen that if (G,E_0) has no 1-, 2-, or 3- split, then (G,E_0) satisfies (iii) or (iv).

Part (2) Let $\mathcal{M}(G,E_0)$ satisfy case (2) of theorem 4.1. The following will be useful in the sequel.

Claim 5. Let G' be an undirected graph, without isolated nodes, such that $\mathcal{M}(G')$ is isomorphic to $\mathcal{M}(G)$. Then G' is isomorphic to G.

Proof of claim 5:
G is 2-connected (since (G,E_0) has no 1-split), so $\mathcal{M}(G)$ is a connected matroid. Since $\mathcal{M}(G)$ is isomorphic to $\mathcal{M}(G')$, G' is 2-connected. Moreover if $\{u,v\}$ is a two node cutset of G, then one side of that cutset consists of two parallel edges only (since there is no 2-split). Now a re-

sult of Whitney's [1933] (cf. Welsh [1976, p. 86]) yields that G is isomorphic to G'.

<u>end of proof of claim 5.</u>

We now consider the three subcases in (2) in Theorem 4.1.

<u>Case I</u>: $M(G,E_0)$ is graphic.
Hence there exists an undirected graph $\tilde{G}$, such that $M(\tilde{G}) \cong M(G,E_0)$. Denote the edge in $E(\tilde{G})$ corresponding to p by e_p. Then $M(G) = M(G,E_0)/p \cong M(\tilde{G})/e_p = M(\tilde{G}/e_p)$. By claim 5 we may assume now: $G = \tilde{G}/e_p$. Taking v_0 equal to the node in which e_p is contracted we obtain that (G,E_0) satisfies (ii).

<u>Case II</u>: $M(G,E_0)$ is cographic. Hence there exists an undirected graph $\tilde{G}$ such that $M^*(\tilde{G}) = M(G,E_0)$. Again, let e_p be the edge in $E(\tilde{G})$ corresponding to p. Then $M(G) = M(G,E_0)/p = M^*(\tilde{G})/e_p = M^*(\tilde{G} \setminus e_p)$. Hence G is planar, and by claim 5 we may assume that $\tilde{G} \setminus e_p$ is its planar dual. The only odd faces of G are the two faces corresponding to the endpoints of e_p in $\tilde{G}$.
<u>Case III</u>: $M(G,E_0) = R_{10}$.
For any element x of R_{10} we have that R_{10}/x is isomorphic to $M^*(K_{33})$. This contradicts the fact that $M(G,E_0)/p = M(G)$ is graphic. So case III cannot occur. □

<u>Remark:</u>
Theorem 4.3 together with Lemma 4.2 yields a polynomial-time algorithm which determines whether or not a given signed graph contains an odd-K_4 or an odd-K_3^2.

5. Orientations and Homomorphisms to odd cycles

An <u>orientation</u> of a signed graph is a replacement of the odd edges by directed edges. If in such an orientation for each cycle the number of forwardly directed edges minus the number of backwardly directed edges is at most k in absolute value, we say that the orientation has <u>discrepancy</u> k.

<u>Theorem 5.1</u>

Let (G,E_0) be a signed graph. (G,E_0) does not contain an odd-K_4 or an odd-K_3^2 if and only if (G,E_0) has an orientation with discrepancy 1.

<u>Proof</u>:

The result follows from the following lemma:

<u>Lemma</u>:

Let M be a $\{0,1\}$- matrix. The matroid $\mathcal{M}$ represented over GF(2) by M is regular if and only if there exists a $\{0, \pm 1\}$-matrix $N \equiv M \pmod 2$ which represents $\mathcal{M}$ over $\mathbb{Z}$.

<u>Proof of the lemma</u>: First we prove the if part.

F_7 and F_7^* are not representable over $\mathbb{Z}$. So by Tutte's characterization of regular matroids (Tutte [1958]) any matroid representable over GF(2) and over $\mathbb{Z}$ is regular.

Next we prove the only if part. (This follows also from the orientability of regular matroids, cf. Welsh [1976, p. 175]. We shall not use this in the proof below.) Let M be partitioned as below, such that M_{11} is a non-singular r x r matrix (over GF(2)), where r is the rank of M over GF(2).

$$\left[\begin{array}{c:c} M_{11} & M_{12} \\ \hdashline M_{21} & M_{22} \end{array}\right]$$

Let M_{11}^{-1} be the matrix inverse of M_{11} over GF(2). Then $\mathcal{M}$ is represented over GF(2) by

(*) $$[I \vdots M_{11}^{-1}M_{12}]$$

(where I denotes the r x r identity matrix). Since M is regular, and (*) is a standard matrix representation of M (cf. Welsh [1976, p. 137]), there exists a $\{-1,0,1\}$matrix $R \equiv M_{11}^{-1} M_{12}$, such that R is totally unimodular, i.e. all subdeterminants of R are 0 or $\pm$ 1 (Tutte [1958]). Moreover $[I \vdots R]$ represents M over $\mathbb{Z}$. Using Ghouila-Houri's characterization of totally unimodular matrices (Ghouila-Houri [1962]), one can prove that there exist $\{-1,0,1\}$ matrices $N_{11} \equiv M_{11}$ (mod 2), $N_{21} \equiv M_{21}$ (mod 2) such that both $N_{11}R$ and $N_{21}R$ are $\{-1,0,1\}$ matrices. N_{11} is nonsingular over Q, since $\det N_{11} \equiv \det M_{11} \equiv 1$ (mod 2), and $N_{11}R \equiv M_{11}R \equiv M_{12}$ (mod 2), and $N_{21}R \equiv M_{21}R \equiv M_{21}M_{11}^{-1}M_{12} \equiv M_{22}$ (mod 2) ($M_{22} \equiv M_{21}M_{11}^{-1}M_{12}$, since M_{11} is of full rank in M). So the desired matrix N equals:

$$\left[\begin{array}{c|c} N_{11} & N_{11}R \\ \hline N_{21} & N_{21}R \end{array}\right]$$

<u>end of proof of lemma</u>.

To prove the theorem, we only consider the only if part. (The if part is trivial.) So, assume (G,E_0) does not contain an odd-K_4 or an odd-K_3^2. Let M be the representation matrix of $M(G,E_0)$ defined in (2.1).

Since $M(G,E_0)$ is regular, the matrix N, as meant in the lemma, exists. We may assume:

$$N = \left[\begin{array}{c|c} 1 & \chi_{E_0} \\ \hline 0 & N^1 \end{array}\right] \qquad \text{where } N^1 \equiv M_G \text{ (mod 2)}$$

(as we may multiply columns by -1). Now N^1 represents the cycle matroid of G over $\mathbb{Z}$.

<u>Claim</u>: We may assume that each column of N^1 has one 1 and one -1.

<u>Proof of the claim</u>: Indeed, take any spanning forest F in G. By multiplying some of the rows of N^1 by -1, we can achieve that each column of N^1 corresponding to the edges in F contains one 1 and one -1. Now the sum of the components of each of these columns is zero. Since F is a basis in $M(G)$, each column of N^1 is a linear combination of the columns corresponding to the edges in F. So each column of N^1 is a linear combi-

nation of the columns corresponding to the edges in F. Hence in each column of N^1 the sum of the components is zero.
Since each column has exactly two nonzero entries, both from $\{1,-1\}$, this proves our claim.

<u>end of proof of claim.</u>

Next we define the orientation: Edge $uv \in E_0$ is directed from u to v if the component of column corresponding with edge uv, indexed by u, v respectively, is -1, 1 respectively. To show that this orientation has discrepancy 1, take any cycle C in (G,E_0). Since (G,E_0) is regular there exists a vector $x = (x_p, x^1) \in \{0,1,-1\}^{\{p\} \cup E(G)}$ such that

(i) $x_e^1 = \pm 1$ if and only if $e \in C$,

(ii) $x_p = \pm 1$ if and only if C is odd,

(iii) $N^1 x^1 = 0$.

From $x_p + \chi_{E_0} x^1 = 0$ one now easily derives that the orientation defined above has discrepancy 1. □

<u>Remark:</u>
Theorem 5.1 can also be proved using Theorem 4.3. We leave this to the reader as an exercise.

The orientation Theorem 5.1 for signed graphs which do not contain an odd-K_4 or an odd-K_3^2 has some interesting applications. These applications will be the content of the remainder of this section. In these applications the following will play a central role: Let (G,E_0) be a signed graph with no odd-K_4 and no odd-K_3^2. Take any orientation of (G,E_0) with discrepancy 1. Orient the edges not in E_0 arbitrary. The set of arcs obtained in this way will be denote by $\vec{A}$. Let $\overleftarrow{A} := \{\overrightarrow{vu} \mid \overrightarrow{uv} \in \vec{A}\}$.

First we shall see that the min-max relations in Corollary 3.2 are quite easily proved for signed graphs with no odd-K_4 and no odd-K_3^2.

<u>Shortest odd cycle</u>
Let $w: E(G) \rightarrow \mathbb{Z}_+$. The shortest odd cycle problem is:

(5.2) Find an odd cycle C in (G,E_0), which minimizes $\sum_{e \in E(c)} w(e)$

If $V \subset V(G)$, we define $[V]$ to be the set of even edges in $E(G)$ leaving V together with the odd edges in $E(G)$ contained in V or in $V(G) \setminus V$. The collection $\{[V] \mid V \subset V(G)\}$ is contained in the collection of subsets of $E(G)$ meeting each odd cycle in (G,E_0). Moreover the edge minimal members of $\{[V] | V \subset V(G)\}$ are exactly the edge minimal subsets of $E(G)$ meeting each odd cycle. Therefore Corollary 3.2 states that if (G,E_0) has no odd-K_4 and no odd-K_3^2, then the minimum value in (5.2) equals the maximum value of the following packing problem:

(5.3) Find a maximum cardinality w-packing of $E(G)$ by sets of the form $[V]$ $(V \in V(G))$.

In order to prove this min-max relation, we consider the following optimization problem (with $\vec{A}$ and $\overleftarrow{A}$ as above)

(5.4) maximize σ

s.t.: There are $\pi_v \in \mathbb{Q}$ for $v \in V(G)$, $\sigma \in Q$, such that for each $\overrightarrow{uv} \in \vec{A}$:

$$|\pi_v - \pi_u + \sigma| < w(uv) \text{ if } uv \in E_0$$
$$|\pi_v - \pi_u| < w(uv) \text{ if } uv \notin E_0$$

For each $\sigma > 0$ we define the following weight function, w^σ, on $\vec{A} \cup \overleftarrow{A}$:

if $a \in \vec{A}$ and a comes from $e \in E_0$: then $w^\sigma(a) := w(e) - \sigma$,
if $a \in \overleftarrow{A}$ and a comes from $e \in E_0$, then $w^\sigma(a) := w(e) + \sigma$,
if a comes from an edge $e \notin E_0$, then $w^\sigma(a) := w(e)$.

It is not hard to see that (5.4) is equivalent to

(5.5) maximize σ

s.t.: There exists no directed cycle in $(V, \vec{A} \cup \overleftarrow{A})$ such that its weight with respect to w^σ is negative.

From the fact that the orientation of (G,E_0), has discrepancy 1 one easily derives that the maximum value, σ^* say, of (5.5) is equal to the minimum value of (5.2). Hence σ^* is integral (since w is an integer weight function). For each $u \in V(G)$, π_u^* is defined as the minimal weight, with respect to w^{σ^*}, of any directed path in $(V, \vec{A} \cup \overleftarrow{A})$ with end-point u. So π_u^* is an integer for each $u \in V(G)$. Moreover, σ^* and π_u^* $(u \in V(G))$ satisfy the constraints in (5.4). Define, for each $i=1,\ldots,\sigma^*$ the sets

$$Z_i := \{z \in \mathbb{Z} \mid z = i+1,\ i+2,\ldots,i+\sigma^* (\mathrm{mod}\ 2\sigma^*)\},$$

and the sets

$$V_i := \{u \in V(G) \mid \pi_u^* \in Z_i\}.$$

Then $\{[V_1],\ [V_2],\ldots,[V_{\sigma^*}]\}$ is a w-packing of E(G). Indeed, this follows easily from the following three

(i) $uv \in [V_i] \cap E_0$ if and only if $|\{\pi_u^*,\ \pi_v^* + \sigma^*\} \cap Z_i| = 1$,

(ii) $uv \in [V_i] \setminus E_0$ if and only if $|\{\pi_u^*,\ \pi_v^*\} \cap Z_i| = 1$,

(iii) for $z_1,\ z_2 \in \mathbb{Z}$:

$$\{i=1,\ldots,\sigma^* \mid |\{z_1,z_2\} \cap Z_i| = 1\} \leq \min\{|z_1-z_2|,\ \sigma^*\}.$$

<u>Conclusion:</u>

If σ^* is the minimum weight of an odd cycle in (G,E_0) then there exists a w-packing of the edges in G by σ^* sets of the form [V] with $V \subset V(G)$. So the min-max relation in Corollary 3.2 (ii) holds for signed graphs with no odd-K_4 and no odd-K_3^2.

<u>Remarks</u>

(i) There exist polynomial algorithms which solve (5.2) (in any signed graph) (Grötschel, Pulleyblank [1981]: Gerards, Schrijver [1985]). For graphs with no odd-K_4 and no odd-K_3^2 the discussion above yields an easy polynomial time algorithm for solving problem (5.3), at least as soon as the orientation with discrepancy 1 is known. Indeed, first find the minimal length, σ^* say, of an odd cycle in (G,E_0). Define the weight func-

tion $w^{\sigma*}$ on the arcs as above. By calculating distances in this weighted directed graph one finds the values π_u^* $(u \in V(G))$. To find the w-packing, some care is needed as σ^* need not be polynomial in the input size. By reducing the values π_u^* $(u \in V(G))$ modulo $2\sigma^*$, we can determine (in polynomial time): $D := \{d \mid 0 \leq d \leq 2\sigma^* - 1$, there exist a $u \in V(G)$: $\pi_u^* \equiv d \pmod{2\sigma^*}\}$. For each $i=0,\ldots,\sigma^*-1$ define $D_i :=$ $\{d \in D \mid i \leq d \leq i + \sigma^* - 1\}$. Now note that in general several of these D_i 's are equal. Instead of determinating all D_i, we determine all sets $\tilde{D}_k$ for which there exists an i with $D_i = \tilde{D}_k$, and the number λ_k of indices i such that $\tilde{D}_k = D_i$. It is not hard to see that this can be done in polynomial time (there are at most $|V(G)|$ of these sets $\tilde{D}_k$). Now the elements of the w-packing will be the sets $[\tilde{V}_k]$ taken with multplicitly λ_k, where
$\tilde{V}_k = \{u \in V \mid$ there exists a $d \in \tilde{D}_k$ such that $\pi_u^* \equiv d \pmod{2\sigma}\}$.

(ii) The dual of the linear program (5.4) is:

(5.6) Minimize $\sum_{a \in \vec{A} \cup \overleftarrow{A}} w(a)f(a)$

s.t. f is a nonnegative circulation in $(V, \vec{A} \cup \overleftarrow{A})$ such that

$$\sum_{a \in \vec{A} \cap E_0} f(a) - \sum_{a \in \overleftarrow{A} \cap E_0} f(a) = 1.$$

It can be shown that (5.6) has an integral optimal solution, and that (5.6) is a reformulation of (5.2). ($a \in \vec{A} \cap E_0$ ($\overleftarrow{A} \cap E_0$) means $a \in \vec{A}$ ($\overleftarrow{A}$ respectively), and a comes from an odd edge.)

<u>Packing with odd cycles</u>

Let $w: E(G) \to \mathbb{Z}_+$. The w-packing problem for odd cycles is:

(5.7) Find a maximum cardinality w-packing of E(G) by odd cycles.

Corollary 3.2 states that, if (G,E_0) has no odd-K_4 and no odd-K_3^2, then the maximum value in (5.7) equals the minimum value in:

(5.8) Find a set $V \subset V(G)$ that minimizes $\sum_{e \in [V]} w(e)$.

($[V]$ is defined in the subsection "shortest odd cycle" of this section). Using the orientation Theorem 4.1 we now shall prove this min-max relation for signed graphs with no odd-K_4 and no odd-K_3^2. Consider the following circulation problem:

(5.9) maximize $\sum_{a \in \vec{A} \cap E_0} f(a) - \sum_{a \in \overleftarrow{A} \cap E_0} f(a)$

s.t. f is a nonnegative circulation $(V, \vec{A} \cup \overleftarrow{A})$, such that:
for each $a_1 \in \vec{A}$, $a_2 \in \overleftarrow{A}$ coming from the same edge $e \in E(G)$: $f(a_1) + f(a_2) \leq w(e)$.

Formulated this way, (5.9) is not a proper circulation problem. However it can be transformed into a circulation problem, as follows: replace each pair $a_1 \in \vec{A}$, $a_2 \in \overleftarrow{A}$ coming from an edge $e \in E(G)$ by the configuration in figure 10. To arc $\vec{e}$ we assign capacity $w(e)$, all other new capacities are ∞.

figure 10

So we see that the maximum in (5.9) is achieved by an integral f.

<u>Lemma 5.10</u>: (5.7) and (5.9) are equivalent.

<u>Proof</u>:

For each cycle C in (G, E_0) we define the circulation f_C as follows. In $\vec{A} \cup \overleftarrow{A}$ there are two directed cycles which correspond in a natural way with C. In case C is odd select from those two cycles that one which uses more edges form $\vec{A}$ then from $\overleftarrow{A}$. In case C is even select an arbitrary one of these two directed cycles. Call the directed cycle chosen D_C. Now let $f_C(a) = 1$ if $a \in D_C$, $f_C(a) = 0$ else. Since orientation $\vec{A}$ has discrepancy 1 we have: $\sum_{a \in \vec{A} \cap E_0} f_C(a) - \sum_{a \in \overleftarrow{A} \cap E_0} f_C(a)$ is equal to 1 if C is odd, and is equal to 0 if C is even. Now, let $C_1,\ldots,C_t$ be a w-packing by odd cycles. Then $f_{C_1} + \ldots + f_{C_t}$ is a feasible solution of (5.9), with objective value t. Conversely, let f be an integral feasible solution of (5.9). Then f is the sum of characteristic vectors of directed cycles in $D(G,E_0)$. The number of odd cycles used in this sum is at least the objective value of f. By the feasibility of f, these odd cycles form a w-packing of E(G). □

The dual linear program of (5.9) is:

(5.11) minimize: $\sum_{e \in E(G)} w(e)\delta(e)$

s.t. $\delta(uv) \in \mathbb{Q}_+$ for $uv \in E(G)$, such that there are $\pi_u \in \mathbb{Q}$ for $u \in V(G)$ satisfying: for each $\overrightarrow{uv} \in \vec{A}$:

$$1-\delta(uv) \leq \pi_v - \pi_u \leq 1 + \delta(uv) \text{ if } uv \in E_0,$$
$$-\delta(uv) \leq \pi_v - \pi_u \leq \delta(uv) \text{ if } uv \notin E_0,$$

Above we have seen that the dual linear program of (5.11), i.e. (5.9), has an integral optimal value for each $w: E(G) \longrightarrow \mathbb{Z}_+$. Hence, so has (5.11). From this it follows that (5.11) has an integral optimal solution. This is a consequence of Lemma 5.12 below. As we shall see, Lemma (5.12) is a corollary of a well known result of Edmonds and Giles [1977] (cf. Schrijver, Corollory 22.1a [1986]).

<u>Lemma 5.12</u>: Let $M \in \mathbb{Z}^{k\times m}$, $N \in \mathbb{Z}^{k\times n}$, and $b \in \mathbb{Z}^k$, such that for each $c \in \mathbb{Z}^k$, for which

(*) $\max \{c^T x \mid Mx + Ny \leq b\}$

exists, the optimal value of (*) is an integer. Then the optimal value of (*) is attained by an $(x,y) \in \mathbb{Z}^m \times \mathbb{Q}^n$ for each c. If moreover, N is totally unimodular, then (*) is attained by an $(x,y) \in \mathbb{Z}^m \times \mathbb{Z}^n$ for each c for which (*) exists.

Proof:

Let $P := \{x \in \mathbb{Q}^n \mid \exists_{y \in \mathbb{Q}^m} [Mx+Ny \leq b]\}$. Then, under the assumptions of the lemma, $\max \{c^T x \mid x \in P\}$ is an integer for each $c \in \mathbb{Z}^n$ (if the maximum exists). From the above mentioned result of Edmonds and Giles [1977], it then follows that P = convex hull $(P \cap \mathbb{Z}^n)$. This settles the lemma. □

For each δ: $E(G) \longrightarrow \mathbb{Q}_+$ we define the weight function $\vec{\delta}$: $\vec{A} \cup \overleftarrow{A} \longrightarrow \mathbb{Q}$ by

$$\vec{\delta}(a) = \begin{cases} \delta(e) + 1 & \text{if } a \in \vec{A}, \text{ and } a \text{ comes from } e \in E_0 \\ \delta(e) - 1 & \text{if } a \in \overleftarrow{A}, \text{ and } a \text{ comes from } e \in E_0 \\ \delta(e) & \text{if } a \in \vec{A} \cup \overleftarrow{A}, \text{ and } a \text{ comes from } e \notin E_0. \end{cases}$$

Obviously (5.11) is equivalent to:

(5.13) minimize $\sum_{e \in E(G)} w(e)\delta(e)$

s.t. $\delta(e) \in \mathbb{Q}_+$ for each $e \in E(G)$, such that there exists no directed cycle in $\vec{A} \cup \overleftarrow{A}$ with negative weight with respect to $\vec{\delta}$.

Lemma: (5.13), and hence (5.11), has a $\{0,1\}$-valued optimal solution δ.

Proof: Orientation $\vec{A}$ has discrepancy 1. Hence for each directed cycle $\vec{C}$ in $\vec{A} \cup \overleftarrow{A}$ (corresponding to cycle C in G) we have that $\sum_{a \in \vec{C}} \vec{\delta}(a) - \sum_{e \in E(C)} \delta(e) = 0, 1$ or -1. Together with Lemma 5.12 this proves the lemma. □

Now, let δ^*, π^* be an integral optimal solution of (5.11) with δ^* $\{0,1\}$- valued. Define $V := \{u \in V(G) \mid \pi^*_u$ is even $\}$. It is straightforward to check that $\delta(uv) = 1$ if and only if $uv \in [V]$. So the optimal solution of (5.11) corresponds with the optimal solution of (5.8).

Conclusion:

If (G, E_0) has no odd-K_4 and no odd-K_3^2, then the maximum of (5.7) equals the mimimum of (5.8). So the min-max relation in Corollary 3.2 (i) holds for signed graphs with no odd-K_4 and no odd-K_3^2.

Next we give another application of the orientation theorem 5.1.

Homomorphisms to odd cycles

Let G_1 and G_2 be undirected graphs. We call a map ϕ: $V(G_1) \longrightarrow V(G_2)$ a homomorphism from G_1 to G_2, if $\phi(u)\phi(v) \in E(G_2)$ for each $uv \in E(G_1)$. A parity preserving subdivision of a signed graph (G,E_0) is an undirected graph, obtained from G by replacing odd (even) edges by paths of odd (even) length. The following result is another characterization of signed graphs with no odd-K_4 and no odd-K_3^2.

Theorem 5.14

Let (G,E_0) be a signed graph. Then (G,E_0) has no odd-K_4 and no odd-K_3^2 if and only if for each parity preserving subdivision G^1 of (G,E_0) with shortest odd cycle C^1, there exists a homomorphism ϕ from G^1 to C^1.

Proof:

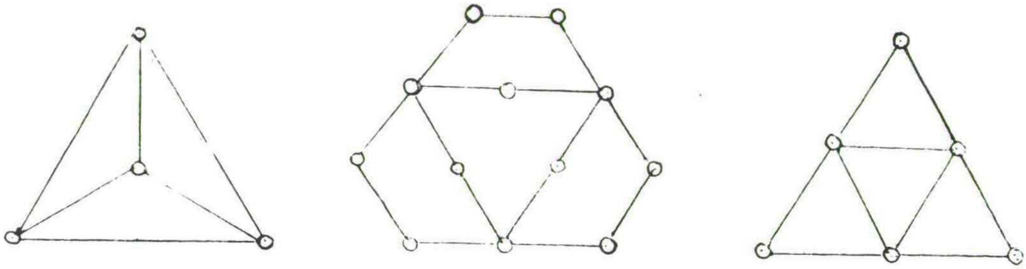
We leave the if part to the reader. E.g. for the graphs in figure 11a, b there exists no homomorphism to their shortest odd cycle. (However, for the graph in figure 11c such homomorphism exists.) For the only if part we may assume: $E_0 = E(G)$, $G^1 = G$. Let the length of the shortest odd cycle in G be $2k + 1$. We define the following weightfunction w: $\vec{A} \cup \overleftarrow{A} \to \mathbb{Z}$

$$w(a) := \begin{cases} k + 1 & \text{if } a \in \vec{A} \\ - k & \text{if } a \in \overleftarrow{A} \end{cases}$$

Using the fact that orientation $\vec{A}$ has discrepancy 1, it is not hard to see that $\vec{A} \cup \overleftarrow{A}$ has no directed cycle with negative weight (with respect to w). Hence there exists a "potential" ϕ: $V(G) \longrightarrow \mathbb{Z}$ satisfying: $\phi_u - \phi_v \leq w(\overrightarrow{uv})$ if $\overrightarrow{uv} \in \vec{A} \cup \overleftarrow{A}$. So ϕ satisfies:

$$k \leq \phi_u - \phi_v \leq k + 1$$

if $\overrightarrow{uv} \in \vec{A}$. Hence for each $uv \in E(G)$: $2\phi_u - 2\phi_v \equiv \pm 1 \pmod{2k + 1}$. So $u \longrightarrow 2\phi(u) \pmod{2k + 1}$ maps G to an cycle of lenght $2k + 1$. □



(a) (b) (c)

figure 11

Remarks:

(i) The proof above relies on Theorem 5.1, and hence on Tutte's deep result that a matroid is regular if and only if it has no F_7 and no F_7^* minor. A direct elementary, though more complicated, proof of (5.14) can be found in Gerards [1985].

(ii) Theorem (5.14) can be used to prove the min-max relation (ii) of lemma 3.1 for signed graphs with no odd-K_4 and no odd-K_3^2 and weight functions w which satisfy: w(e) is odd if and only if $e \in E_0$.

(iii) From theorem 5.2 we immediately get: Let G be an undirected graph, with no odd-K_4 and no odd-K_3^2 (E_0=E(G)). Then G is 3-colorable. This is a special case of a result of Catlin [1979] (Theorem 6.3 of this paper). Using a similar technique as in the proof of (5.14) one can prove the following result of Minty [1962]: A graph G has an orientation such that for each cycle C the number of forward edges with respect to each of both orientations, of C is at least $\frac{1}{k}$ $|E(C)|$ if and only if G is k-colorable.

Indeed, "only if" is trivial; "if" follows similarly to the proof of Theorem (5.14) by defining:

$$w(a) := \begin{cases} k-1 & \text{if } a \in \vec{A} \\ -1 & \text{if } a \in \overleftarrow{A} \end{cases}$$

(iv) Theorem 5.14 extends a result of Albertson, Catlin, and Gibbons [1984] stating that an undirected graph G can be mapped homomorphically onto an odd cycle of lenght M if no subgraph of G can be fold to a homeomorf of K_4 in which all faces are cycles of lenght M (<u>folding</u> means iteratively identifying nodes at distance two).

<u>Stable sets</u>

A <u>stable set</u> in an undirected graph G is a subset S of V(G), such that $uv \notin E(G)$ for each $u, v \in S$.
The <u>stable set polyhedron</u>, $P_S(G)$, of G is the convex hull of the characteristic vectors of all stable sets in G. Using Theorem 5.1 one can prove:

<u>Theorem 5.15</u>:
Let G be an undirected graph, containing no odd-K_4 and no odd-K_3^2 (E_0 = E(G)). Then the system of inequalities:

$$\begin{cases} x_u \geq 0 & (u \in V(G)) \\ x_u + x_v \leq 1 & (uv \in E(G)) \\ \sum\limits_{u \in V(C)} x_u \leq \frac{|V(C)|-1}{2} & (C \text{ odd cycle in } G) \end{cases}$$ □

is a so called <u>totally dual integral</u> system for $P_S(G)$. (cf. Edmonds, Giles [1977]).

This result can be extended to graphs with no odd-K_4, hereby extending a result of Boulala and Uhry [1979]. (Gerards [1986], forthcoming paper.)

6 Signed Graphs with no odd-K_4

Signed graphs with no odd-K_4 have interesting properties with respect to combinatorial optimization. In section 3 we mentioned Seymour's result (Lemma (3.2), (i)). A second one is the following:

Let A be an integral m × n matrix such that in each row the sum of the absolute values of the entries is at most 2. Define the signed graph Σ(A) as follows. Delete the rows which have one non-zero entry. Consider the new matrix as the edge-node incidence matrix of a graph G (rows correspond to edges). The edges corresponding to rows with a 1 and a -1 will be even in Σ(A) , the other edges odd.

Theorem 6.1 (Gerards, Schrijver [1985])

Let A be an integral m × n matrix, such that in each row the sum of the absolute values of the entries is at most 2. Then the following are equivalent:

(i) $\Sigma(A)$ does not contain an odd-K_4.

(ii) For each a, b $\in \mathbb{Z}^n$; c, d $\in \mathbb{Z}^m$ the convex hull of all integral vectors in

$$P := \{x \in \mathbb{Q}^n \mid a \leq x \leq b;\ c \leq Ax \leq d\}$$

is equal to the intersection of the halfspaces $\{x \in \mathbb{Q}^n \mid cx \leq [\beta]\}$, where $c \in \mathbb{Z}^n$, $\beta \in \mathbb{Q}$ such that $cx \leq \beta$ for each $x \in P$. ($[\beta]$ denotes the largest integer not greater than β.) □

The following two theorems on signed graphs with no odd-K_4 are proved using Theorem 2.6 and results from the previous sections for signed graphs with no odd-K_4 and no odd-K_3^2.

First we state a decomposition theorem, due to Lovász, Schrijver, Seymour, and Truemper [1984, unpublished paper]. It immediately follows from Theorems 2.6 and 4.3.

Theorem 6.2

Let (G,E_0) be a signed graph containing no odd-K_4. Then (G,E_0) has a 1-, 2-, or 3- split, each part of the split contains no odd-K_4, or (G,E_0) is a bipartite signed graph with one extra node (and edges joining that node), or (G,E_0) is planar with at most two odd faces, or $(G,E_0) = K_3^2$, or (G,E_0) is the signed graph of figure 8. □

Next we prove a result of Catlin [1979] using Theorems 2.6 and 5.1.

Theorem 6.3 (Catlin [1979])

Let G be an undirected graph. If $(G,E(G))$ does not contain an odd-K_4 then G is 3-colorable.

Proof:

Let G be a minimal counterexample. Obviously G is 2-connected. Suppose $\{u,v\}$ is a two node cutset of G. Then, one part of this cutset (possibly after adding an edge from u to v) is a smaller counterexample. So G is

3-connected, and by theorem 2.6 it contains no odd-K_3^2. Now theorem 5.14 yields that G is 3-colorable, a contradiction. $\square$

7 Grafts, T-joins

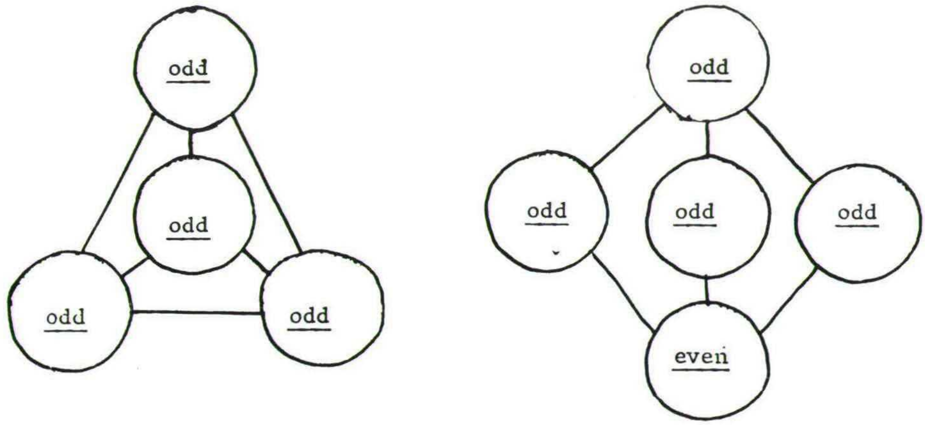
Along the lines of the previous sections we state some results on an object called graft, by Seymour [1980]. A graft is a pair [G,T], where G is an undirected graph and T a subset of V(G). Associated with a graft we define the following binary matroid $M[G,T]$. Let M_G be the node-edge incidence matrix of G. Moreover let $\chi_T \in \mathbb{R}^{V(G)}$ be the characteristic vector of T as a subset of V(G). Then $M[G,T]$ is the binary matroid represented over GF(2) by:

$$\left[\begin{array}{c|c} M_G & \chi_T \end{array}\right]$$

The element of $M[G,T]$ corresponding to the last column of this matrix will be denoted by t. A T-join is a collection E' of edges, in E(G) such that each $v \in T$ meets an odd number of edges in E^1, and each $v \notin T$ meets an even number of edges in E^1. The circuits of $M[G,T]$ are the cycles in G, and all unions of $\{t\}$ with a minimal T-join in G. If $V \subset V(G)$ such that both $V \cap T$ and $(V(G) \setminus V) \cap T$ are odd then the collection, $\delta(V)$, of edges from V to $V(G) \setminus V$ is called a T-cut. Note that the minimal T-cuts are exactly those minimal edge sets meeting each T-join. Conversely the minimal T-joins are the minimal edge sets meeting each T-cut.

Remark:
There is a similarity between grafts and signed graphs. Take an arbitrary minimal T-join E_0 in G. Then the circuits of $M^*[G,T]$ are the even cuts, and each union of $\{t\}$ with an odd cut. Here odd (even) means, containing an odd (even) number of edges from E_0: so $M[G,T]$ is obtained from $M^*(G)$ by signing in the same way as $M(G,E_0)$ is obtained from $M(G)$.

We define two special types of grafts: a K_4-partition and a $K_{3,2}$-partition. They are indicated in figure 12. Circles stand for connected subgraphs, odd (even) indicates that the corresponding connected subgraph contains an odd (even) number of members of T, and lines stand for edges.



K_4-partition $\qquad$ $K_{3,2}$-partition

Figure 12

In case each circle contains exactly one point we use the terms: the graft K_4, the graft $K_{3,2}$ respectively. I.e. the graft K_4 is $[K_4, V(K_4)]$, where K_4 is the 4-clique, the graft $K_{3,2}$ is $[K_{3,2},T]$ where $K_{3,2}$ is the complete bipartite graph with colorclasses of size 3 and 2, and $T = V(K_{3,2}) \setminus \{w\}$ where w is one of the nodes of degree 3. We say that a graft $[G,T]$ contains a K_4-partition ($K_{3,2}$-partition) if each component of G contains a even number of points in T, and at least one component contains a K_4-partition ($K_{3,2}$-partition respectively) covering that component. (By covering we mean that each node of the component is a node of the K_4-partition ($K_{3,2}$-partition respectively).

We also define reduction operations for grafts. There are: deletion of an edge, and contraction of an edge. In the latter case we have to modify T too. If edge uv is contracted into the new node w, then T/uv is $T \setminus \{uv\}$ if $|\{u,v\}| \cap T|$ is even and $(T\setminus\{u,v\}) \cup \{w\}$ else. If the graft $[G_2,T_2]$ is obtained from the graft $[G_1, T_1]$ by one or more of these reductions we say: $[G_1,T_1]$ reduces to $[G_2,T_2]$. The relation with matroid minors is obvious:

- $\mathcal{M}[G,T]\setminus e = \mathcal{M}[G\setminus e,T]$:
- $\mathcal{M}[G,T]/e = \mathcal{M}[G/e,T/e]$.

Moreover

- $\mathcal{M}[G,T]\backslash t = \mathcal{M}(G)$:
- $\mathcal{M}[G,T]/t$ is the binary matroid with as circuits: all minimal T-joins and all cycles not containing a T-join.

The following is easy to prove.

<u>Lemma 7.1.</u>

Let $[G,T]$ be a graft. Then the following are equivalent:

(i) $\mathcal{M}[G,T]$ has an F_7-minor using t;

(ii) $[G,T]$ reduces to the graft K_4;

(iii) $[G,T]$ contains a K_4-partition.

Similarly, the following are equivalent:

(i) $\mathcal{M}[G,T]$ has an F_7^*-minor using t:

(ii) $[G,T]$ reduces to the graft $K_{3,2}$:

(iii) $[G,T]$ contains a $K_{3,2}$-partition. □

<u>Corollary 7.2.</u>

Let $[G,T]$ be a graft. Then $\mathcal{M}[G,T]$ is regular if and only if $[G,T]$ does not contain an K_4-partition or a $K_{3,2}$-partition. □

<u>Min-max relations</u>

Like in section 3, from Seymour's characterization of matroids with the max-flow-min-cut-property (Seymour [1976]), the following result follows:

<u>Theorem 7.3.</u>

Let $[G,T]$ be a graft. Then the following are equivalent:

(i) $[G,T]$ contains no K_4-partititon;

(ii) For each weight function $w: E(G) \to \mathbb{Z}_+$ the minimum weight of a T-join equals the maximum cardinality of a w-packing with T-cuts.

Similarly, the following are equivalent:

(i)' $[G,T]$ contains no $K_{3,2}$-partition;

(ii)' For each weight function $w: E(G) \to \mathbb{Z}_+$ the minimum weight of a T-cut equals the maximum cardinality of a w-packing with T-joins.

<u>Decompositions</u>

Now we go along the lines of section 4. First we define the notion of <u>splits</u> for grafts. Assume E_1 and E_2 are non-empty subsets of $E(G)$, partitioning E(G). Denote the set of nodes in V(G) spanned by E_1, E_2 by V_1, V_2, respectively. $\bar{G}_1$ is defined by: $V(\bar{G}_i) := V_i$, $E(\bar{G}_i) := E_i$ $(i=1,2)$. Moreover assume $|T|$ even and non-zero.

1-split

If $|V_1 \cap V_2| = 0$, then $[\bar{G}_1, T_1 \cap V_1]$, $[\bar{G}_2, T \cap V_2]$ is a 1-split of $[G,T]$. If $|V_1 \cap V_2| = 1$, $V_1 \cap V_2 = \{u\}$ say, and G_1 and G_2 are connected, then $[\bar{G}, T_1]$, $[\bar{G}_2, T_2]$ is a 1-split of $[G,T]$. T_1 is defined as $T \backslash V_2$ if $|T \cap V_2|$ is even, and as $(T \backslash V_2) \cup \{u\}$ if $|T \cap V_1|$ is odd. T_2 is defined similarly.

2-split

If $|V_1 \cap V_2| = 2$, $V_1 \cap V_2 = \{u,v\}$ say, and $\bar{G}_1$ and $\bar{G}_2$ are connected, then we define $[G_1, T_1]$ as follows.

If $T \backslash V_1 = \emptyset$ then $V(G_1) := V_1$, $E(G_1) := E_1 \cup \{uv\}$, and $T_1 := T$.

If $T \backslash V_1 \neq \emptyset$, then $V(G_1) := V_1 \cup \{v^*\}$, $E(G_1) = E_1 \cup \{uv^*, v^*v\}$, and $T_1 := (T \cap V_1) \cup \{v^*\}$ if $|T \backslash V_1|$ is odd, $T_1 := (T \cap V_1) \Delta \{u, v^*\}$ if $|T \backslash V_1|$ is even. $[G_2, T_2]$ is defined similarly. The pair $[G_1, T_1]$, $[G_2, T_2]$ obtained in this way is called a 2-split, unless $\bar{G}_1$ or $\bar{G}_2$ is equal to the graph in figure 13 below, and $w \in T$.

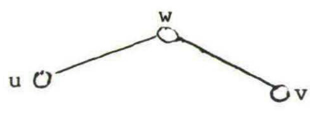


figure 13

3-split:

If $|V_1 \cap V_2| = 3$, $V_1 \cap V_2 = \{u_1, u_2, u_3\}$ say, $\bar{G}_1$ and $\bar{G}_2$ are connected, and $T \subset V_1$, $|E_2| > 4$, then we define $[G_1, T_1]$ as follows. $V(G_1) := V_1 \cup \{v^*\}$, $E(G_1) := E_1 \cup \{u_1v^*, u_2v^*, u_3v^*\}$, and $T_1 := T$. We call $[G_1, T_1]$ a 3-split. (A 3-split has one part only.)

The following is straightforward to prove

Lemma 7.4.

Let $[G,T]$ be a graft with a k-split ($k \leq 3$) and no ℓ-split for any $\ell < k$. Then $[G,T]$ has no K_4-partition and no $K_{3,2}$-partition if and only if each part of the k-split has no K_4-partition and no $K_{3,2}$-partition.

Proof:

Under the conditions mentioned each part of a split is a reduction of the original graft. This settles one side of the equivalence. The other side can be proved by case checking. □

Now we state and prove a decomposition result for grafts with no K_4-partition and no $K_{3,2}$-partition.

Theorem 7.5

Let $[G,T]$ be a graft containing no K_4-partition and no $K_{3,2}$-partition. Then one of the following holds:

(i) $[G,T]$ has a 1-, 2-, or 3-split.

(ii) $|T|$ is odd or $|T| \leq 2$.

(iii) G is planar with all members of T on one common face.

(iv) $G = K_{3,3}$, and $T = V(K_{3,3})$.

Proof:

If $[G,T]$ has no 1- or 2-split, then $M[G,T]$ is graphic if and only if (ii) holds, $M[G,T]$ is co-graphic if and only if (iii) holds, and $M[G,T] = R_{10}$ if and only if (iv) holds. The proofs are similar to Part (2) of the proof of Theorem 4.3.

The assumptions imply that $M[G,T]$ is regular. Assume $[G,T]$ has no 1-, 2- or 3-split and does not satisfy one of (ii), (iii) and (iv). We are going to derive a contradiction. By Theorem 4.1 we have a partition $E_1 \cup E_2$ of $E(G)$ such that

$$(*) \qquad r_{M[G,T]}(E_1) + r_{M[G,T]}(E_2 \cup \{t\}) = r_{M[G,T]}(E(G) \cup \{t\}) + k - 1$$

with $k = 1, 2$ and $|E_1|$, $|E_2| + 1 \geq k$.

or $k = 3$ and $|E_1|$, $|E_2| + 1 \geq 6$.

For each $E' \subset E(G)$ we have:

$$r_{M[G,T]}(E') = r_{M(G)}(E') = r_{M[G,T]}(E' \cup \{t\}) - \varepsilon(E')$$

where $\varepsilon(E') = 0$ if each component of $(V(G),E')$ spans an even number of points in T, and $\varepsilon(E') = 1$ else.
So from (*) we get:

$$(**) \qquad r_{M(G)}(E_1) + r_{M(G)}(E_2) = r_{M(G)}(E(G)) + (k-\varepsilon) - 1,$$

where $\varepsilon := \varepsilon(E_2)$ ($\varepsilon(E(G)) = 0$, since, if not, then G is disconnected, or $|T|$ is odd).
Define $E_1^1,\ldots,E_1^s$, $E_2^1,\ldots,E_2^t$, and the auxilary graph H as in the proof of Theorem 4.3 (Note, that if $E_2 = \emptyset$, then $k = 1$ and $\varepsilon = 0$. So $T = \emptyset$, and (ii) holds).

Claim 1: H is a bipartite connected graph with no isthmuses. Moreover $|E(H)| = |V(H)| + k - \varepsilon - 2$.
Proof of claim 1:
The proof is similar to the proofs of claim 1 and 2 of the proof of Theorem 4.3.

end of proof of claim 1

Claim 2: $k = 3$, $\varepsilon = 0$: H is homeomorf to the graph in figure 14(b).
Proof of claim 2: If H is a cycle, then [G,T] would have a 2-split. Claim 1 now yields $k - \varepsilon - 2 > 1$. So $\varepsilon < k - 3$, i.e. $k = 3$, $\varepsilon = 0$. So $|E(H)| = |V(H)| + 1$. Since H has no isthmuses, H is homeomorf to one of the graphs in figure 14. If H is homeomorf with the graph in figure 14(a), then [G,T] has a 2-split. So H is homeomorf with the graph in figure 14(b).

end of proof of claim 2

(a) (b)

figure 14

Hence G is of the form as in figure 15, where $A, B \in \{E_1^1, \ldots E_1^s, E_2^1, \ldots, E_2^t\}$, and C_1, C_2 and C_3 are unions of elements of $\{E_1^1, \ldots, E_1^s, E_2^1, \ldots, E_2^t\} \setminus \{A,B\}$. Note that for $i = 1, 2, 3$ it is possible that $u_i = v_i$, so $C_i = \emptyset$.

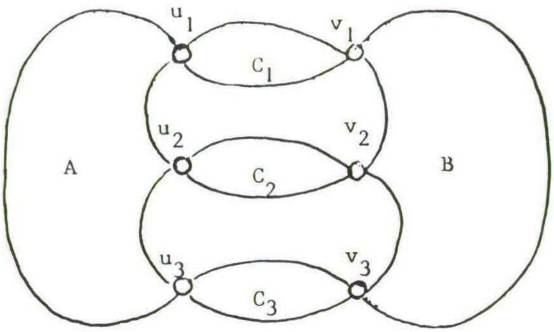


figure 15

<u>Claim 3</u>: $C_i = \emptyset$, $C_i = \{u_iv_i\}$, or $C_i = \{u_iw_i, w_iv_i\}$ with $w_i \in T$, for $i = 1, 2, 3$. Moreover $|C_1| + |C_2| + |C_3| < 5$.

<u>Proof of claim 3</u>: The first part of the claim follows since [G,T] has no 2-split. If the second part would not be true, then $C_i = \{u_iw_i, w_iv_i\}$ with $w_i \in T$ for each $i = 1, 2, 3$. But then [G,T] has a $K_{3,2}$-partition (T is even), a contradiction.

<u>end of proof of claim 3</u>

<u>Claim 4</u>: $A \cup B = E_1$, $C_1 \cup C_2 \cup C_3 = E_2$.

<u>Proof of claim 4</u>: Since $|E_1| > 6$, E_1 cannot be contained in $C_1 \cup C_2 \cup C_3$. So we may assume $A = E_1^1$. The edges in $C_1 \cup C_2 \cup C_3$ which are adjacent with u_1, u_2, or u_3 cannot be in E_1 (Since A is a component of E_2). Now from claim 3 and, again, $|E_1| > 6$ if follows that $B = E_1^2$. Since $E_2 > 5$, and $|C_1| + |C_2| + |C_3| < 5$: $C_1 \cup C_2 \cup C_3 = E_2$

<u>end of proof of claim 4</u>

Claim 5: G is the graph in figure 16; w_1, $w_3 \in T$.
Proof of claim 5: From the previous it follows that we only need to prove that $A = E_1^1$ and $B = E_1^2$ (figure 15) are triangles. If $|E_1^1|$ or $|E_1^2|$ is greater than or equal to 4, [G,T] has a 3-split. Since $|E_1| > 6$, this yields $|E_1^1| = |E_1^2| = 3$. If E_1^1 or E_1^2 is not a triangle then one easily finds a 1- or 2-split.

end of proof of claim 5

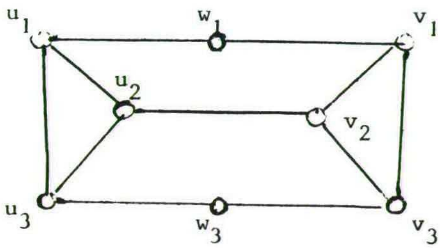


figure 16

So w_1, $w_3 \in T$. If $u_2 \in T$, or $v_2 \in T$, then we would have a $K_{3,2}$-partition (as $|T|$ is even). Hence T lies on the outer face of the planar graph G, i.e. (iii) holds, a contradiction. □

Orientations

The following result is proved similarly as Theorem 5.1.

Theorem 7.6

Let [G,T] be a graft. [G,T] has no K_4-partition and no $K_{3,2}$-partition if and only if one of the following holds.

(i) $|T|$ is odd.

(ii) There exists a partition T_1, T_2 of T with $|T_1| = |T_2|$ such that each T-join is an edge disjoint union of cycles and of $|T_1|$ paths from T_1 to T_2. □

Remark: Theorem 7.6 yields the following result (answering a question of A. Frank).

Let G be an undirected connected graph. Then the following are equivalent:

(i) G has an orientation $\vec{A}$ such that
$|\{\overrightarrow{uv} \in \vec{A} | u \in X,\ v \notin X\}| - |\{\overrightarrow{uv} \in \vec{A} | u \notin X,\ v \in X\}| \leq 1$, for each minimal cut $\delta(X)$.

(ii) $[G, V_{odd}]$ has no K_4-partition and no $K_{3,2}$-partition.
(V_{odd} denotes the set of nodes in G with odd degree.)

We shall only indicate how (i) follows from (ic). Assume (ii) holds. Let T_1, T_2 be a partition of V_{odd} as in meant in (7.6 (ii)). Let $C_1, \ldots C_k$, be a collection of cycles in G, and $P_1, \ldots, P_{|T_1|}$ a collection of paths from T_1 to T_2 such that $E(C_1), \ldots, E(C_k)$, $E(P_1), \ldots, E(P_{|T_1|})$ partition E(G) (E(G) is a V_{odd}-join). Now orient G such that each C_i becomes a directed cycle, and each P_i becomes a path directed from its endpoint in T_1 to its endpoint in T_2. That this orientation satisfies (i) follows from the observation that if $\delta(X)$ is a minimal cut then $|X \cap T_1| - |X \cap T_2| \leq 1$. (Indeed, since $\delta(X)$ is a minimal cut, there exists a V_{odd}-join F such that $|F \cap \delta(X)| \leq 1$. Applying 7.6 (ii) to F yields $|X \cap T_1| - |X \cap T_2| \leq 1$.)

Using Theorem 7.6 we shall now prove the min-max relations in Lemma 7.2 for the case that [G,T] has no K_4-partition and no $K_{3,2}$-partition. So let [G,T] have this property. We may assume, that G is connected and $|T|$ is even. Let T_1, T_2 be a partition of T as is meant in Theorem 7.6. We define a directed graph D as follows. V(D) = V(G), and the arcset A of D is obtained by replacing each edge uv by two arcs, one $\overrightarrow{uv}$, from u to v, the other, $\overrightarrow{vu}$ from v to u.

Shortest T-join

Let $w: E(G) \to \mathbb{Z}_+$. The shortest T-join problem is:

(7.7) Find a T-join $E' \subset E(G)$, which minimizes $\sum_{e \in E'} w(e)$.

This optimization problem is equivalent to the following circulation problem ($w(a) := w(uv)$ for $a = \overrightarrow{uv}$ or $a = \overrightarrow{vu}$ with $uv \in E(G)$).

$$(7.8)\quad \text{minimize} \sum_{a \in A} w(a)f(a)$$

$$\text{s.t.} \quad \sum_{a \text{ enters } u} f(a) - \sum_{a \text{ leaves } u} f(a) = \begin{cases} 1 & \text{if } u \in T_1 \\ -1 & \text{if } u \in T_2 \\ 0 & \text{if } u \in V(G) \setminus T \end{cases}$$

$$f(a) > 0 \quad \text{if } a \in A.$$

To prove the equivalence, first observe that any T-join E' in $E(G)$ is the edge disjoint union of $|T_1|$ paths from T_1 to T_2 and, possibly, some cycles. So there exists a feasible solution of (7.8), with $\sum_{a \in A} w(a)f(a) = \sum_{e \in E'} w(e)$. Conversely, let $f\colon A \to \mathbb{Q}_+$ be an optimal solution of (7.8). Since the constraint matrix of (7.8) is totally unimodular we may assume that $f(a) \in \mathbb{Z}$ for each $a \in A$. The set of arcs $E' := \{a \in A \mid f(a) \text{ is odd}\}$ is a T-join, with $\sum_{e \in E'} w(e) < \sum_{a \in A} w(a)f(a)$. So (7.7) and (7.8) are equivalent.

The dual linear program of (7.8) is (7.9) below; again there are integral optimal solutions.

$$(7.9)\quad \text{Maximize} \sum_{u \in T_1} \pi_u - \sum_{u \in T_2} \pi_u$$

$$\text{s.t. } \pi_v - \pi_u < w(\overrightarrow{uv}) \text{ if } \overrightarrow{uv} \in A.$$

Equivalently:

$$(7.10)\quad \text{maximize} \sum_{u \in T_1} \pi_u - \sum_{u \in T_2} \pi_u$$

$$\text{s.t. } |\pi_v - \pi_u| < w(uv) \qquad \text{if } uv \in E(G).$$

Let $\pi \in \mathbb{Z}^{V(G)}$ be an optimal solution of (7.10). Define for each λ with $\min\{\pi_u \mid u \in V(G)\} < \lambda < \max\{\pi_u \mid u \in V(G)\}$, the set $V_\lambda := \{u \in V(G) \mid \pi_u > \lambda\}$. We shall need the following lemma.

Lemma 7.11 Let $U \subset V(G)$ such that the subgraph of G induced by U is connected. Then $\delta(U)$ contains at least $||U \cap T_1| - |U \cap T_2||$ mutually edge-disjoint T-cuts.

Proof: Let $V_1,\dots,V_\ell$ be the node sets of the components of the subgraph of G induced by $V(G)\setminus V$. Let, without loss of generality, $V_1,\dots,V_k$ $(k \leq \ell)$ be those sets V_i with $|V_i \cap T|$ odd. Take edges $e_1,\dots,e_k \in E(G)$ from $V_1,\dots,V_k$, respectively, to U. Then there exists a T-join E' such that each $e \in E'$ is entirely contained in V, or in V_i $(i=1,\dots,\ell)$, or is an element of $\{e_1,\dots,e_k\}$. Since E' contains an edge disjoint union of $|T_1|$ paths from T_1 to T_2 it follows that $k \geq ||V \cap T_1| - |V \cap T_2||$. Since each $\delta(V_i)$, $i=1,\dots,k$, is a T-cut this proves the lemma. □

Using this lemma we can construct a w-packing with T-cuts of cardinality at least $\sum_{u \in T_1} \pi_u - \sum_{u \in T_2} \pi_u$. For each $\lambda \in \mathbb{Z}$ and each component V of V_λ such that $|V \cap T_1| - |V \cap T_2| > 0$, take $|V \cap T_1| - |V \cap T_2|$ mutually edge disjoint T-cuts in $\delta(V)$. The T-cuts obtained in this way from the desired T-packing. Indeed, they form a w-packing since the sets $\delta(V_\lambda)$ do so. Moreover the cardinality of this w-packing is greater than or equal to $\sum_{u \in T_1} \pi_u - \sum_{u \in T_2} \pi_u$ (since the components V of V_λ with $|V \cap T_1| - V \cap T_2| \leq 0$ are not used to construct the w-packing). What we have proved now is that the minimum of (7.8) is not greater than the maximum value in the following packing problem:

(7.12) Find a maximum cardinality w-packing with T-cuts.

The fact that this maximum is not smaller than the minimum of (7.8) is trivial. Hence we have proved the min-max relation (ii) in Lemma 7.3 for signed graphs with no K_4-partition and no K_3^2- partition.

Packing with T-joins

Let $w \in \mathbb{Z}^{E(G)}$. Consider the problem:

(7.13) Find a maximum cardinality w-packing with T-joins

We shall prove that (7.13) is equivalent to

(7.14) maximize k

$$\text{s.t.} \quad \sum_{a \text{ enters } u} f(a) - \sum_{a \text{ leaves } u} f(a) = \begin{cases} k \text{ if } u \in T_1 \\ -k \text{ if } u \in T_1 \\ 0 \text{ if } u \in V(G)\backslash T \end{cases}$$

$$f(\overrightarrow{uv}) + f(\overrightarrow{vu}) < w(uv) \text{ if } uv \in E(G)$$
$$f(\overrightarrow{uv}) > 0 \qquad \text{if } \overrightarrow{uv} \in A.$$

The fact that each w-packing with k T-joins yields a feasible solution of (7.14) of value k is obvious. Conversely, let $f^*: A \to \mathbb{Q}_+$, $k^* \in \mathbb{Q}_+$ form an optimal solution of (7.14) which is not a convex combination of other optimal solutions.

Lemma 7.15 $k^* \in \mathbb{Z}_+$: $f^*(a) \in \mathbb{Z}_+$ for $a \in A$.

Proof Obviously, if k^* is integer an then so is $f^*(a)$ for $a \in A$. (Observe the construction in figure 10.) Assume $k^* \notin \mathbb{Z}_+$. Let E' be the set of edges $uv \in E(G)$ for which $0 < f(\overrightarrow{uv}) + f(\overrightarrow{vu}) < w(uv)$. Let $V_1, \ldots, V_\ell$ be the vertex sets of the components of E'. If E' would contain a T-join, then f^*, k^* would not be optimal. Let $E^0 \subset E(G)\backslash E'$ be a minimal set so that $E^0 \cup E'$ contains a T-join. Then there exists a set V_{i^*} ($i^*=1,\ldots,\ell$) such that there is exactly one edge, e say, in E^0 leaving V_{i^*}. Let F be a minimal T-join in $E^0 \cup E'$. By the minimality of E^0 the edge e must be in F. Since F is the edge-disjoint union of $|T_1|$ paths from T_1 to T_2 we now know that $|V_{i^*} \cap T_1| - |V_{i^*} \cap T_2| = \pm 1$. Now the fact that $f^*(\overrightarrow{uv})$, $f^*(\overrightarrow{vu}) \in \mathbb{Z}$ for each $uv \in \delta(V_{i^*})$, and the fact that f^* and k^* form a feasible solution to (7.14) contradicts the fact that $k^* \notin \mathbb{Z}$. □

Next we must prove that there exists a w-packing with T-joins, of cardinality k^*. This follows (by induction) from the following lemma:

Lemma 7.16: Let $k \in \mathbb{Z}_+$, $f \in \mathbb{Z}_+^A$ be a feasible solution of (7.14), with $k > 1$. Then there exists a solution $k_1 \in \mathbb{Z}_+$, $f_1 \in \mathbb{Z}_+^A$ with: $k_1 = 1$, and for each $a \in A$ if $f_1(a) > 0$ then $f(a) > 0$.

Proof: Define the following capacitated auxilary digraph D'. $V(D') = V(D) \cup \{s,t\}$. (s and t are two new nodes). The arc set A(D') of D' consists of the arcs in A together with all arcs of the form $\overrightarrow{su}$ with $u \in T_2$ and $\overrightarrow{ut}$ with $u \in T_1$. The capacity function $c: A(D') \to \mathbb{Z}_+$ is defined by: $c(a) = f(a)$ if $a \in A$, $c(\overrightarrow{su}) = 1$ if $u \in T_2$, and $c(\overrightarrow{ut}) = 1$ if $u \in T_1$. If the lemma is not true then the maximal flow from s to t in this capacitated auxilary digraph is less than $|T_2|$. By the max-flow-min-cut theorem there exists a $U \subset V(G)$ such that

$$\sum_{\substack{a \in A(D') \\ a \text{ leaves } U \cup \{s\}}} c(a) < |T_2|.$$

Hence

$$(*) \qquad \sum_{\substack{a \in A \\ a \text{ leaves } U}} f(a) + |T_2 \setminus U| + |T_1 \cap U| < |T_2|.$$

Since f and k form a feasible solution of (7.14) we have

$$\sum_{\substack{a \in A \\ a \text{ leaves } U}} f(a) \geq \max\{0, k|T_2 \cap U| - k|T_1 \cap U|\}$$

Combining this with (*) we get

$$\max\{0, k|T_2 \cup U| - k|T_1 \cup U|\} < |T_2 \cup U| - |T_1 \cup U|.$$

which contradicts with $k \geq 1$. □

The dual linear program of (7.14) is

$$(7.17) \quad \text{minimize} \sum_{e \in E(G)} w(e)\ell(e)$$

$$\text{s.t.} \quad \pi_v - \pi_u + \ell(uv) \geq 0 \text{ if } \overrightarrow{uv} \in A$$

$$\ell(e) \geq 0 \text{ if } e \in E(G)$$

$$\sum_{u \in T_1} \pi_u - \sum_{u \in T_2} \pi_u = 1.$$

For each $w \in \mathbb{Z}_+^{E(G)}$ the minimum value of (7.17) is an integer. (By Lemma 7.15 and linear programming duality.) Hence, by Lemma 5.12, (7.17) has an integral optimal solution.

Using this lemma we can prove that (7.17) is equivalent to:

(7.18) Find a $U \subset V(G)$ with $|V(G) \cap T|$ odd, such that $\sum_{e \in \delta(U)} w(e)$ is minimum.

To prove the equivalence, first assume that π, ℓ form an integral feasible optimal solution of (7.17). Then there exists $\lambda \in \mathbb{Z}$ such that $\tilde{V} := \{u | \pi_u = \lambda\}$ satisfies $|\tilde{V} \cap T_1| \neq |\tilde{V} \cap T_2|$ (since $\sum_{u \in T_1} \pi_u - \sum_{u \in T_2} \pi_u = 1$). Since each T-join contains $|T_1|$ mutually edge disjoint paths from T_1 to T_2, $\delta(\tilde{V})$ contains a T-cut, $\delta(U)$ say. Moreover for each $uv \in \delta(\tilde{V})$: $\pi_u - \pi_v \neq 0$; so $\ell(uv) \geq 1$. Therefore

$$\sum_{e \in \delta(U)} w(e) \leq \sum_{e \in \delta(\tilde{V})} w(e) \leq \sum_{e \in E(G)} w(e)\ell(e).$$

Conversely let U be an optimal solution to (7.18). By Lemma 7.11 we may assume that $|U \cap T_1| - |U \cap T_2| = 1$. Define $\pi_u := 1$ if $u \in U$; $\pi_u := 0$ if $u \in V(G) \setminus U$; $\ell(e) = 1$ if $e \in \delta(U)$ and $\ell(e) = 0$ if $e \in E(G) \setminus \delta(U)$. Then

$\sum_{e \in \delta(U)} w(e) = \sum_{e \in E(G)} w(e)\ell(e)$ and π and ℓ form a feasible solution of (7.17).

Grafts with no K_3^2-partition.

The following result is of the same nature as Theorem 2.6.

Theorem 7.19

Let [G,T] be a graft with no K_4-partition. If G has no one node cutset, and for each two node cutset $\{u,v\}$ of G, one side of the cut consists of two edges uv^* and v^*v in series, with $v^* \in T$, then [G,T] has no $K_{3,2}$-partition, or [G,T] is equal to the graft $K_{3,2}$ (i.e G is the bipartite graph $K_{3,2}$; $T = V(G) \setminus \{w\}$, where w has degree 3).

Proof:

Assume [G,T] satisfies the assumptions and contains a $K_{3,2}$-partition. We shall prove that [G,T] equals the graft $K_{3,2}$. First we define extended $K_{3,2}$-partition, by figure 17. The sets U_1, U_2, V_1, V_2, V_3, W_1, W_2, W_3 cover V(G). The graphs induced by these sets are connected. For each i=1,2,3 $|V_i \cap T|$ is odd, and $|W_i \cap T|$ is odd or $W_i = \emptyset$. The lines are edges.

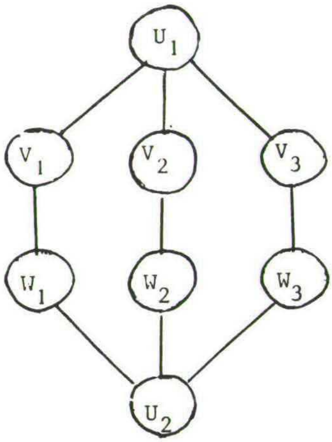


figure 17

Since [G,T] has a $K_{3,2}$-partition, it has an extended $K_{3,2}$-partition. Let U_1, U_2, etc.... be an extended $K_{3,2}$-partition with $|U_1| + |U_2|$ minimal. First note that if there would exist an edge from $V_i \cup W_i$ to $V_j \cup W_j$ $(i \neq j)$ then [G,T] would have a K_4-partition.

Claim 1: There exists a $u_1 \in U_1$ and edges from u_1 to V_1, V_2, and V_3. Also there exists a $u_2 \in U_2$ and edges from u_2 to W_i or if $W_i = \emptyset$ to V_i for i = 1, 2, 3.

Proof of claim 1: Obviously, we only need to prove the existence of u_1. There exists a node $u \in U_1$ and mutually node disjoint paths P_1, P_2, P_3 from u to V_1, V_2, V_3 respectively, such that the only points of P_1, P_2, P_3 not in U_1 are the end points in V_1, V_2, V_3 respectively. Let X be the set of nodes in U_1 which lie on P_2 or on P_3 (so $u \in X$). Denote by U_1' the set of nodes $v \in U_1 \setminus X$ for which there exists a path in $U_1 \setminus X$ from v to V_1. We prove that $U_1' = \emptyset$, i.e. P_1 consists of a single edge. By symmetry between P_1, P_2 and P_3 also P_2 and P_3 are single edges, so the node $u_1 \in U_1$ exists: indeed, take $u_1 = u$. Hence we may suppose $U_1' \neq \emptyset$. We shall construct an extended $K_{3,2}$-partition, contradicting the minimality of $|U_1| + |U_2|$. Replace U_1 by $U_1 \setminus U_1'$. If $|U_1' \cap T|$ is even replace V_1 by $U_1' \cup V_1$. If $|U_1' \cap T|$ is odd and $W_1 \neq \emptyset$ replace V_1 by $U_1' \cup V_1 \cup W_1$, and W_1 by the empty set. If $|U_1' \cap T|$ is odd and $W_1 = \emptyset$ replace V_1 by U_1', and set $W_1 := V_1$. All other sets in the original $K_{3,2}$-partition remain the same. By this we obtained a new $K_{3,2}$-partition violating the assumed minimality of $|U_1| + |U_2|$.

<u>end of proof of claim 1</u>

<u>Claim 2</u>: $|U_1| = |U_2| = 1$

<u>Proof of Claim 2</u>: Let $\tilde{U}_i := U_i \setminus \{u_i\}$ $(i=1,2)$ where u_1 and u_2 are the nodes established by claim 1. As in the proof of claim 1 one easily shows that no edge leaving $\tilde{U}_1$ or $\tilde{U}_2$ can enter $\bigcup_{i=1}^{3} (V_i \cup W_i)$. Assume $\tilde{U}_1 \cup \tilde{U}_2 \neq \emptyset$. Then $\{u_1, u_2\}$ is a two node cutset of G since there are no edges from $V_i \cup W_i$ to $V_j \cup W_j$ $(i \neq j)$. We may assume that the two parts of G separated by $\{u_1, u_2\}$ are $V_1 \cup V_2 \cup W_1 \cup W_2$ and $V_3 \cup W_3 \cup \tilde{U}_1 \cup \tilde{U}_2$. Since they both do not correspond to two edges in series we have a contradiction. So $\tilde{U}_1 \cup \tilde{U}_2 = \emptyset$, and claim 2 is settled.

<u>end of proof of claim 2</u>

Since there are no edges from $V_i \cup W_i$ to $V_j \cup W_j$ for $i \neq j$ $(i,j=1,2,3)$, the condition on two nodes cutsets of G yields that [G,T] is the graft $K_{3,2}$ (note that since T is even exactly one of u_1, u_2 is in T). □

Theorems 7.5 and 7.19 yield a decomposition result for grafts with no K_4-partition. With arguments similar to the remark at the end of section 2 one can prove that if G is 3-connected and has no $K_{3,2}$-partition then it has no K_4-partition or is equal to the graft K_4 (with $T=V(K_4)$).

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